Образцы экзаменационных задач по дисциплине «Уравнения математической физики» для потока К-6 (годовой курс)

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I) Решить методом Фурье задачи для уравнения теплопроводности:

1) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = x(1-x), \end{cases} \qquad u = u(x,t) = ?$$

2) $$\begin{cases} u_t = 4u_{xx}, \quad 0 < x < 2\pi, \quad t > 0, \\ u(0,t) = 0, \quad u(2\pi,t) = 0, \\ u(x,0) = 1, \end{cases} \qquad u = u(x,t) = ?$$

3) $$\begin{cases} u_t = u_{xx} - u, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \pi - x, \end{cases} \qquad u = u(x,t) = ?$$

4) $$\begin{cases} u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \cos^2 x + \cos^2 3x, \end{cases} \qquad u = u(x,t) = ?$$

5) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = 16\cos^4 x, \end{cases} \qquad u = u(x,t) = ?$$

6) $$\begin{cases} u_t = u_{xx} - 2u + \sin\dfrac{x}{2}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \sin\dfrac{3x}{2}, \end{cases} \qquad u = u(x,t) = ?$$

7) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u_x(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 1 - x^2, \end{cases} \qquad u = u(x,t) = ?$$

II) Решить методом Фурье задачи для уравнения колебаний:

1) $\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 1, \end{cases}$ $\qquad u = u(x,t) = ?$

2) $\begin{cases} u_{tt} = u_{xx} - 2u, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = \sin 2x, \quad u_t(x,0) = \sin x, \end{cases}$ $\qquad u = u(x,t) = ?$

3) $\begin{cases} u_{tt} = 4u_{xx} - 4\sin 5x, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

4) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \\ u(x,0) = 1 - x, \quad u_t(x,0) = 1, \end{cases}$ $\qquad u = u(x,t) = ?$

5) $\begin{cases} u_{tt} = u_{xx} - u, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \cos^2 x, \quad u_t(x,0) = \sin^2 x, \end{cases}$ $\qquad u = u(x,t) = ?$

6) $\begin{cases} u_{tt} = u_{xx} - 4t\sin^2 3x, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

7) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \sin \dfrac{3x}{2}, \quad u_t(x,0) = 2\sin \dfrac{x}{2}, \end{cases}$ $\qquad u = u(x,t) = ?$

8) $\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < 2, \quad t > 0, \\ u(0,t) = 0, \quad u_x(2,t) = 0, \\ u(x,0) = x, \quad u_t(x,0) = -x, \end{cases}$ $\qquad u = u(x,t) = ?$

III) Найти непрерывные решения для следующих задач на полупрямой:

1) $$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = t + 1, \\ u(x,0) = 1, \quad u_t(x,0) = x + 1, \end{cases} \qquad u = u(x,t) = ?$$

2) $$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = \sin 5t, \\ u(x,0) = 0, \quad u_t(x,0) = 5\cos 5x, \end{cases} \qquad u = u(x,t) = ?$$

3) $$\begin{cases} u_{tt} = 25u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = \sin 5t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases} \qquad u = u(x,t) = ?$$

4) $$\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = 0, \\ u(x,0) = x^3, \quad u_t(x,0) = 0, \end{cases} \qquad u = u(x,t) = ?$$

5) $$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = 1, \\ u(x,0) = 2x, \quad u_t(x,0) = 0, \end{cases} \qquad u = u(x,t) = ?$$

6) $$\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = \sin x, \end{cases} \qquad u = u(x,t) = ?$$

7) $$\begin{cases} u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases} \qquad u = u(x,t) = ?$$

8) $$\begin{cases} u_{tt} = 100u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = t + 1, \\ u(x,0) = x + 1, \quad u_t(x,0) = x + 1, \end{cases} \qquad u = u(x,t) = ?$$

IV) В круге $x^2+y^2<R^2$ найти гармоническую функцию $u=u(x,y)$, если:

1) $R=5, \qquad u\big|_{x^2+y^2=25}=x^2, \qquad u=u(x,y)=?$

2) $R=4, \qquad u\big|_{x^2+y^2=16}=(x+y)^2, \qquad u=u(x,y)=?$

3) $R=3, \qquad u\big|_{x^2+y^2=9}=x^2-2xy-4y^2, \qquad u=u(x,y)=?$

4) $R=2, \qquad u\big|_{x^2+y^2=4}=1+x+x^2+x^3, \qquad u=u(x,y)=?$

5) $R=1, \qquad u\big|_{x^2+y^2=1}=y^4, \qquad u=u(x,y)=?$

V) В шаре $x^2+y^2+z^2<R^2$ найти гармоническую функцию $u=u(x,y,z)$, если:

1) $R=5, \qquad u\big|_{x^2+y^2+z^2=25}=z(5+z), \qquad u=u(x,y,z)=?$

2) $R=4, \qquad u\big|_{x^2+y^2+z^2=16}=(z+4)^2, \qquad u=u(x,y,z)=?$

3) $R=\dfrac{1}{3}, \qquad u\big|_{x^2+y^2+z^2=\frac{1}{9}}=1+3z+9z^2, \qquad u=u(x,y,z)=?$

4) $R=\dfrac{1}{2}, \qquad u\big|_{x^2+y^2+z^2=\frac{1}{4}}=z(1-4z^2), \qquad u=u(x,y,z)=?$

5) $R=1, \qquad u\big|_{x^2+y^2+z^2=1}=(z+1)^3, \qquad u=u(x,y,z)=?$

VI) В шаровом слое $1<r<2$ пространства $\mathbb{R}^n$ найти гармоническую функцию $u=u(r)$, если:

1) $n=2, \qquad u\big|_{r=1}=1, \qquad u\big|_{r=2}=-2, \qquad u=u(r)=?$

2) $n=3, \qquad u\big|_{r=1}=2, \qquad u\big|_{r=2}=1, \qquad u=u(r)=?$

3) $n=4, \qquad u\big|_{r=1}=1, \qquad u\big|_{r=2}=0, \qquad u=u(r)=?$

4) $n=5, \qquad u\big|_{r=1}=2, \qquad u\big|_{r=2}=\dfrac{1}{2}, \qquad u=u(r)=?$

5) $n=100, \qquad u\big|_{r=1}=1, \qquad u\big|_{r=2}=2, \qquad u=u(r)=?$